The Special Theory of Relativity

1. The laws of nature are the same in all inertial frames of reference

2. The speed of light in the vacuum is the same in all inertial frames of reference
Now suppose that the passenger and the robber agree to perform another experiment, in which the passenger sets off a flashbulb on the floor of the train. The light travels to the ceiling and hits a mirror, which reflects it back to the bulb. Both observers measure the time for the light to make the round trip from the bulb to the mirror and back. For the passenger, this is straightforward; the height of the car is $H$, so the round trip travel time is simply $\Delta t_P = 2H/c$. It is simple for the robber as well, but to him, the entire train moves some distance in the time it takes for the light to reach the ceiling and to return. Let the length of this angled path, from the floor to the ceiling, be denoted by $d$. By the second relativity postulate, the speed of light is the same in the robber’s frame as in the passenger’s, so the travel time he will measure is $\Delta t_R = 2d/c$. Since $d$ is greater than $H$, $\Delta t_R$ must be greater than $\Delta t_P$. What does this mean? We can regard this apparatus, in which light bounces from the ceiling and returns to its source, as a clock. One round trip is one tick of the clock. Our experiment shows that each tick of the clock takes longer, that is, time runs slower, for a clock located in an inertial frame that is moving with respect to some specified observer. This phenomenon is called time dilation.

Figure 7.1 A light flash goes off in the center of a moving train. In the train frame, the light hits the front and back of the train simultaneously. In the ground frame, the train is moving with velocity $v$ so the light hits the rear of the train before reaching the front. Two events that are simultaneous in one frame are not simultaneous in the other.

Figure 7.2 Relativistic time dilation. The path of a flash of light traveling from a bulb, up to a mirror, and back, as seen in (a) the rest frame of the train and (b) the rest frame of someone in the ground frame watching the train go by. In the ground frame the train moves to the right a distance $v\Delta t_R$ during the round trip of the light. The resulting light path is longer for the observer in the ground frame, but since the speed of light is the same in all frames, the time intervals in the two frames must be different.

Just how much slower is each tick of the moving clock? The robber remembers his geometry and uses the Pythagorean theorem to compute the distance $d$ traveled by the light in his frame of reference:

$$d^2 = H^2 + \left(\frac{1}{2}v\Delta t_R\right)^2.$$  \hspace{1cm} (7.2)

Recall that in the robber’s frame, $\Delta t_R = 2d/c$, and in the passenger’s frame $\Delta t_P = 2H/c$, so we can eliminate $d$ and $H$ to obtain a quadratic equation

$$\frac{1}{4}(c\Delta t_R)^2 = \frac{1}{4}(c\Delta t_P)^2 + \frac{1}{4}v^2(\Delta t_R)^2.$$  \hspace{1cm} (7.3)

Working through the algebra leads us to the result

$$\Delta t_R = \frac{\Delta t_P}{\left[1 - (v^2/c^2)\right]^{1/2}}.$$  \hspace{1cm} (7.4)
In equation (7.4) we found that the relationship between the time intervals in the two frames contained the factor

\[
\frac{1}{\sqrt{1 - (v/c)^2}} = \Gamma.
\]

(7.5)

Does this look familiar? It appeared in equation (7.1), the Lorentz contraction. Perhaps we are now beginning to understand its significance. It does not tell us anything about a physical contraction of moving matter, but rather describes the way in which space and time are related for observers who are moving with respect to one another. It is often called the **boost factor** between two inertial frames.

\[ c = 300,000 \text{ km/s} = 670 \text{ million mph} \]
Length Contraction

For example, let us return to the train and train robber, and consider two telephone poles beside the tracks. The robber wishes to measure the distance between the poles, and to do so he will make use of the train, which is traveling at known velocity \( v \). The robber simply measures the time interval required for the front edge of the train to pass from the first pole to the second, \( \Delta t_G \); he thus determines that the distance between the poles must be \( \Delta x_R = v \Delta t_R \). Now suppose the passenger on the train decides to measure the distance between the same two telephone poles, which are moving with respect to him at speed \( v \). Both the passenger and the robber agree on the relative speed \( v \), as they must if inertial frames are to be equivalent. The passenger uses a similar timing technique of noting when the first, and then the second, pole passes the edge of his window; he measures a time interval \( \Delta t_P \) between the passage of the first pole and the second. Thus the distance between the poles in the train frame is \( \Delta x_P = v \Delta t_P \). You can see clearly from this example that space measurements are tied in with time measurements, which should not be surprising since time and space intervals are related by speed or velocity. We have already solved for \( \Delta t_R \) in terms of \( \Delta t_P \); hence we can obtain

\[
\frac{\Delta x_P}{\Delta x_R} = \frac{\Delta t_P}{\Delta t_R} = \left[ 1 - (v^2/c^2) \right]^{1/2}
\]

or

\[
\Delta x_P = \Delta x_R \sqrt{1 - (v^2/c^2)}.
\]

which is exactly the Lorentz-FitzGerald contraction, equation (7.1). Thus we have demonstrated that the passenger measures the distance between the moving poles to be shorter than the distance measured by the robber, in whose frame the poles are at rest. The distance measured by the passenger is specified by the self-same factor that was first proposed as an ad hoc explanation for some unexpected experimental results. Now it appears naturally and elegantly from the two fundamental postulates of special relativity.
Energy - Mass Equivalence

\[ E = m \cdot c^2 \]

The strange velocity addition rule of special relativity hints at another important consequence besides the intermingling of space and time, time dilation, and the Lorentz contraction. It leads us to what is perhaps the most famous equation in history, \( E = m \cdot c^2 \). But what does this renowned equation mean, and how does it fit into relativity theory? First we must specify what we mean by "energy." We have previously defined energy as "the capacity to do work." In the Newtonian universe, energy is not created or destroyed, but only transformed from one form to another. Similarly, there is a separate conservation law for matter; matter is neither created nor destroyed. One of the most important forms of energy is kinetic energy, or the energy of motion. In Newtonian mechanics, it can be shown that the kinetic energy of a particle is given by

\[ E_k = \frac{1}{2} m v^2, \]  
(7.10)

where \( m \) is the mass of the particle, and \( v \) is its speed. The Newtonian kinetic energy of a particle at rest is zero. The Einsteinian equation is the relativistic generalization of this concept of kinetic energy. The equation is more correctly written as

\[ E = \Gamma m_0 c^2, \]  
(7.11)

where \( \Gamma \) is our new acquaintance, the boost factor, and \( m_0 \) is the rest mass of the particle, that is, its mass as measured in its own rest frame. Notice that, since the boost factor is 1 for a particle at rest, this definition of energy does not vanish for \( v = 0 \). Thus we find that in relativity there is a rest energy, given by \( m_0 c^2 \), associated with every massive particle. As the speed of the particle increases, its energy also increases. For small velocities, it is possible to show that the relativistic energy equation reduces to

\[ E = m_0 c^2 + \frac{1}{2} m_0 v^2 + \text{additional terms}, \]  
(7.12)

where if \( v \) is very much less than \( c \), the additional terms are very, very small, and we recover the Newtonian law, with the addition of the new concept of the rest energy. At the other extreme, as the speed increases and begins to approach that of light, the relativistic energy becomes very large, much larger than the simple Newtonian rule would predict; it is arbitrarily large for speeds arbitrarily close to the speed of light.

As an example, consider how much energy would be required to accelerate 1 kg of matter to \( 0.87c \), for which \( \Gamma = 2 \). In order to compute the relativistic kinetic energy, we must subtract from the total energy, as given by equation (7.11), the rest energy specified by \( m_0 c^2 \). Carrying out this procedure, taking care to keep our units consistent, we obtain a result of \( 9 \times 10^{16} \text{ joules} \) (J) of kinetic energy. In units that might be more familiar, this is \( 3 \times 10^{10} \text{ kilowatt-hours} \), or about 20 megatons TNT equivalent. You would need all the energy released by a very large thermonuclear bomb in order to accelerate just 1 kg of matter to a speed close to \( c \). This is a serious limitation on our ability to boost anything, even elementary particles such as protons, to speeds approaching that of light. At accelerator laboratories around the world, scientists do just that, accelerating protons to speeds near light speed. It is no coincidence that very large power lines can be seen going onto the grounds of these accelerators!
The surface of the light cone divides spacetime into distinct regions for all observers. The region inside the cone corresponds to all events that are separated by timelike intervals from event $A$. The region outside the cone contains all those events separated from $A$ by spacelike intervals. The half-cone below $A$ is called the past light cone. The past light cone and the timelike region within it make up the past of event $A$. Similarly, the half-cone above $A$ is called the future light cone of $A$, and this half-cone and the timelike region it encompasses compose the future of $A$. Events outside the light cone of $A$ are in the elsewhere of $A$. Given two events, $B$ within the past light cone of $A$, and $C$ within its future light cone, it can be shown that all observers will agree that $B$ is in the past of $A$ and $C$ is in its future, although they may not agree about where and when within the past and future, respectively, these events occur. Thus all observers agree that $B$ occurs before $C$. On the other hand, for an event $D$ in the elsewhere of $A$, observers may disagree on the order of events $A$ and $D$; some may see that $A$ occurs first, while others may observe that $D$ happens first, and still others may regard the two events as simultaneous.

But if different observers do not agree on the ordering of events that are spacelike separated, are we in danger of losing the idea of cause and effect? Can we find a frame in which the lights go on before the switch is thrown? Most of us believe intuitively that a cause must always precede its effect. This has been formalized into the principle of causality, and it is one of the guiding principles of physics. It cannot be proven from any physical laws; it is, in some respects, one of the axioms of physics. But without it, we cannot make sense of the universe. Science is based on the belief that we can understand the universe; its success at this endeavor is ample demonstration of the power of its axioms.

Figure 7.7: The light (null) cone seen (a) with two spatial and one time dimension, and (b) as a simple $(x, t)$ plot. A light cone can be constructed at each event in a spacetime. The light cone divides spacetime into those events in the past (e.g., event $B$), in the future (e.g., event $C$), and elsewhere (e.g., event $D$).
If spacetime were like our usual \((x, y)\) space coordinates, we would be on familiar ground. Given two points in \((x, y)\) coordinates, we could use the Pythagorean theorem to obtain the distance between them. Recall that the square of the distance along the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides:

\[
(\Delta r)^2 = (\Delta x)^2 + (\Delta y)^2. \tag{7.13}
\]

This Pythagorean formula defines the distance between two points on a flat surface, in terms of perpendicular coordinates \(x\) and \(y\).

But spacetime is not like ordinary space. Given two events which lie close to one another on a world line, how can we define a "distance" between them? And what would that distance mean? Each of the two events occurs at points labeled by appropriate \((x, t)\) coordinates. We define the distance, or \textit{spacetime interval}, between them by

\[
\Delta s = \sqrt{(c\Delta t)^2 - (\Delta x)^2}. \tag{7.14}
\]

The factor of \(c\) ensures that we are subtracting two quantities with the same units. This distance measures separations in what is now known as \textit{Minkowski spacetime}, in honor of Hermann Minkowski, who first formulated its properties. The principal difference between the spacetime distance formula and the Pythagorean formula is the negative sign between the time interval and the space interval. Time may be the fourth dimension, but it is not simply like the three space dimensions. The properties of this spacetime geometry are different from the ordinary flat \textit{Euclidean space} to which we are accustomed.

\[\Delta s=0\] for a photon traveling at the speed of light.
We can illustrate the Twin Paradox on a spacetime diagram. First we must recognize that there are *three* inertial reference frames relevant to this problem. The first is the frame of the stay-at-home twin, Andy. (We will ignore the motions of the Earth, since they are so tiny compared to the speed of light. We will also assume that Alpha Centauri is at rest with respect to Earth, although this is not essential.) The second is the inertial reference frame of Betty while traveling to Alpha Centauri, and the third is her inertial reference frame during her return voyage. Figure 7.8 illustrates the round trip in all three of the inertial frames. Betty departs for Alpha Centauri, traveling at constant, very high velocity, at event 1. She turns around at event 2, reversing her direction and returning at the same high, constant velocity. She arrives home at event 3. Notice that in all three reference frames, Andy's world line is straight, indicating that he remains inertial for the entire trip. In contrast, Betty's world line is not straight in all three diagrams, but changes direction at event 2.

The elapsed proper time along any world line is obtained by summing the spacetime intervals along that world line. Since the paths through spacetime differ, we should not be surprised that the proper times recorded by clocks traveling along different world lines between the same two events (Betty's leaving and returning) will be different. The maximum amount of proper time between any two events is that recorded by a clock that follows the straight line through spacetime between those two events; that is, the clock which remains in one inertial, constant-velocity frame. This means that in Minkowski spacetime, the longest time between any two events is a straight line! The fact that the straight line is a maximum, rather than a minimum, is another consequence of the negative sign in the spacetime interval, and another way in which relativity can confuse us. Euclidean space and Minkowskian spacetime have different properties.

If the maximum proper time is obtained by the inertial clock moving along the straight line in spacetime, which clock would show the minimum time? If you wish to record zero time between two events, there is only one way to do it: follow the light beam. A beam of light sent out into space and bounced back to Earth would follow a noninertial, yet still lightlike world line, and the spacetime interval along any lightlike path, accelerated or not, is always zero. Objects with nonzero mass, such as Betty, cannot travel at the speed of light, but Betty can minimize her proper time between two events by traveling as close as she can to the light cone. In all three frames of the Twin Paradox, only the world line of Andy is a straight line through spacetime. Betty travels from event 1 to event 3 by a noninertial route close to the light cone. Hence her clock reads less elapsed proper time than does Andy's, and the faster Betty travels, the smaller her elapsed proper time. There is no "Twin Paradox" once we understand this.
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Special Relativity describes the relationship between measurements in separate inertial frames.

General Relativity deals with the nature of reference frames.

In an inertial reference frame, one does not experience forces:
- in normal situations on Earth we feel weight, the force of gravity
  --> we are NOT in an inertial reference frame

- in normal situations we are continually being accelerated with respect to an inertial frame; i.e., the tug of gravity toward the center of the Earth is equivalent to the experience of being accelerated away from the Earth.

In Newtonian mechanics: An object that falls toward the Earth is viewed as being accelerated in the reference frame of an observer at a point on Earth.

In General Relativity: A falling object is following a force-free path and the Earthbound observer is accelerating (with the surface of the Earth) in a direction away from the center of the Earth.
Satellites and the Moon are in inertial orbits around the Earth
- The pull of gravity is matched by centrifugal force
  (the satellite/Moon falls toward the Earth but, because it has tangential motion, it misses a collision)
- gravity is equivalent to any other acceleration

weak equivalence principle
strong equivalence principle
- all inertial frames are equivalent

consequences of equivalence:

- **gravitational redshift** - acceleration toward (or away from) a light source causes a blue (or red) shift of the light waves
  acceleration $\leftrightarrow$ gravity
  $\rightarrow$ light propagating toward (away from) a mass is blue (red) shifted

- **gravitational time dilation**
  - clocks in an accelerating frame run slow compared to clocks in an inertial frame
  acceleration $\leftrightarrow$ gravity
  $\rightarrow$ clocks in a strong gravitational field run slower than clocks in a weaker gravitational field
  [similarly there is gravitational length contraction]
Tests of General Relativity

* bending of light - eg, starlight by Sun
* modification of orbits - eg, 43s/century discrepancy of orbit of Mercury
  - orbital decay in binary pulsars
* gravitational redshift & time dilation

moving masses change the curvature of space
  --> waves of curvature
  --> gravitational radiation

Tides are gradients in gravitational forces
  --> because of gradients, inertial frames are local
Gravitational lensing by a cluster of galaxies
gravitational lensing
Gravity Waves
Adding a third observatory in Italy

Observed deformity is 0.0001 x width of a proton !!!