We can construct models of the expansion of the Universe without a complete description of the Universe. We can consider a relatively small piece and it will have a history that duplicates the whole (where 'relatively small' just means big enough to have the average density properties of the Universe as a whole: a 'fair sample').

Consider spherical shells of uniform density and consider the force on an observer from an external shell:

**Birkhoff's Theorem**

\[
\text{force on observer due to chunk of the shell: } \text{force } \propto \frac{\text{mass}}{\text{distance}^2}
\]

\[
\text{mass} = \text{density} \times \text{volume}
\]

\[
\text{volume} = \text{base} \times \text{height} \times \text{thickness}
\]

\[
\text{projection angle} \times \text{distance}
\]

Combining:
\[
\text{volume} = (\text{projection angle} \times \text{distance})^2 \times \text{thickness}
\]

So:
\[
\text{force } \propto \text{density} \times (\text{projection angle}^2 \times (\text{distance})^2 \times \text{thickness})
\]

(distance) cancel top & bottom \Rightarrow force only depends on the density + thickness of the shell and the projection angle of the chunk.
Birkhoff's Theorem continued ...

But this is a shell external to the observer, so there are contributions from every direction that the observer looks. For every force within a projection angle in one direction there is a force in the opposite direction. Since the forces are independent of distance, the forces are equal and cancel.

Therefore there is no net force on the observer from an external shell.

There is force on the observer from shells inside his/her radius because the forces come from only certain directions and not others.

The essential reason for the cancellation is that force from a chunk on the shell depends on distance squared but the volume of the chunk increases as the surface area which also depends on distance squared. [We are considering uniform density. Lumps will cause forces that are not cancelled.]

Conclusion: Our model of the Universe can be a uniform sphere with a discrete radius. The forces from uniformly distributed matter outside this radius cancel and can be ignored.
Friedmann Equation

Expansion velocities from distant parts of the Universe become so fast that they approach the speed of light. However, we can avoid problems of relativity by just considering a small ‘fair sample’ sphere of the Universe.

The kinetic energy of expansion of this sphere + the potential energy associated with the mutual attraction of the matter within the sphere = a constant

\[
\frac{1}{2} V^2 - \frac{GM}{R} - \frac{\Lambda R^2}{6} = -\frac{kc^2}{2}
\]

Define

Hubble Constant = \( H = V/R \) = expansion velocity / size

Mass = \( M = (4\pi/3)R^3\rho \) where \( \rho \) = density

Substituting our definitions:

\[
\frac{1}{2} H^2 R^2 - \frac{G\rho 4\pi}{3} R^3 - \frac{\Lambda R^2}{6} = -\frac{kc^2}{2}
\]

So:

\[
H^2 - \frac{8\pi}{3} G\rho - \frac{\Lambda}{3} = -\frac{kc^2}{R^2}
\]

\( k \) is the space curvature constant:

\( k = +1 \) spherical geometry (closed universe)
\( k = 0 \) flat geometry
\( k = -1 \) hyperbolic geometry (open universe)

In the case with no vacuum energy (\( \Lambda = 0 \)), if KE dominates PE then \( k < 0 \) - open universe
If PE dominates KE then \( k > 0 \) - closed universe
The in-between case, \( k = 0 \), specifies the ‘Einstein – de Sitter’ case. The mass density just balances the KE of expansion. We call This the ‘critical density’, \( \rho_{crit} \)

\[
\rho_{crit} = \frac{3 H^2}{8\pi G}
\]
different possible universes

q = deceleration parameter

If no dark energy (\(\Lambda=0\)) then
q = (density)/(2xcritical density)
= 1/2 in flat universe

Figure 11.5  The behavior of the scale factor for a variety of models, all constrained to pass through the time now with the same slope. The value of the deceleration constant determines both the model’s future and the age of the universe. The larger the value of \(q\), the shorter the time back to the big bang. The exponentially expanding de Sitter model, with \(q = -1\), never intersects \(R = 0\).
Fundamental parameters that describe the expanding Universe

1. Time since Big Bang: \( t \)
2. Temperature: \( T \)
3. Density of matter: \( \rho_m \)
4. Scale length or dimension: \( R \)

Let's see how these properties (parameters) are interrelated. That is, if we know one property, how do we then know another?

1. The change of matter density with scale length.

   Mass is conserved in a fixed region expanding with the universe.

   Mass in a spherical volume = density x volume
   \[ = \rho_m \times \frac{4}{3} \pi R^3 \]

   Conservation of mass \( \Rightarrow \rho_m R^3 = \text{constant} \)

   Here: \( \rho_m \propto \frac{1}{R^3} \)

   \( \rho_m \propto \frac{1}{R^3} \)

   \( 1 \rightarrow 2^3 = 8 \) Double size of universe \( \rightarrow \) density down by \( \frac{1}{8} \)
Fundamental parameters (continued)

Change of energy density with scale length (or radius)

Number of photons conserved

⇒ number density of photons vary with scale length the same way as matter

\[ n_y \propto \frac{1}{R^3} \]

However, the energy contained in each photon is falling as the universe expands.

\[ \lambda \rightarrow \lambda^2 \rightarrow 1/\lambda \rightarrow 1/\lambda^2 \]

as universe doubles then wavelength of photon doubles; ie energy \( \propto 1/\lambda^2 \)

ie, energy/photon \( \propto 1/R \)

energy density = photon number density \( \propto 1/R^3 \) \( \rightarrow \) energy per photon \( \rightarrow 1/R \)

\[ \rho_y \propto \frac{1}{R^4} \]

ie, double size of universe ⇒ energy density drops a factor \( 2^4 = 16 \)
C. Change of temperature with scale length.

Temperature is characterized by the Planck distribution of black body radiation

\[ \lambda_{\text{max}} = \frac{0.279 \hbar c}{kT} \]

or \[ T \propto \frac{1}{\lambda_{\text{max}}} \]

But we just saw that \[ \lambda_{\text{max}} \propto R \]

\[ T \propto \frac{1}{R} \]

i.e., double universe then double \( \lambda \) so also double \( T \)
Fundamental parameters (continued)

\[ H^2 = \frac{8\pi G}{3} \rho \]

Consider the simple case where kinetic energy balances potential energy (the "Einstein-De Sitter case with \( k = 0 \) is a "flat" universe with critical density).

Then the Friedman eqn becomes:

\[ H^2 = \frac{8\pi G}{3} \rho \]

or

\[ t = \sqrt{\frac{3}{8\pi G \rho}} \]

If the universe is radiation dominated (i.e., \( \rho_g \gg \rho_m \))

then since \( \rho_g \propto 1/R^4 \), we find:

\[ t \propto \sqrt{R} \]

or

\[ R \propto t^{1/2} \]

If the universe is matter dominated (i.e., \( \rho_g \ll \rho_m \))

then since \( \rho_m \propto 1/R^3 \), we find:

\[ t \propto \sqrt{R^3} \]

or

\[ R \propto t^{2/3} \]

Finally, an example of another case is if the density of the universe is negligible, whence the Friedmann eqn reduces to \( H^2 = -k/R^2 \) where \( k < 0 \) (open expanding universe).

In this case, \( t^2 \propto R \), or \( R \propto t \)

* The case of INFLATION is another interesting example, but we will look at that later.
density vs. lookback time
temperature vs. lookback time

$\Lambda$ becomes important

$t \sim 400,000$ yrs

$t \sim 40,000$ yrs
temperature and density with lookback

Extrapolate today’s CMB density and temperature values back in time using the relations A-D described in previous slides. Kinks in the projected curves occur at transition between matter dominated (today) and radiation dominated (early on). This transition is the consequence of the different dependencies of matter and radiation density with the expansion.

Recombination (surface of last scattering) occurred when the Universe was 400,000 years old, at a lookback redshift of about 1000. In these plots I have the kink between the radiation and matter dominated eras only shortly before. We now think this transition occurred at a lookback redshift of about 3000 when the Universe had an age of about 40,000 years.

At earlier times, the interesting moments were the first few minutes and earlier when temperatures and densities were comparable and above the conditions in the cores of stars today. These were the moments of the Big Bang.

In summary, the temperatures and densities as a function of time are well-specified by our current measurements of the cosmic microwave background radiation. We are able to make this translation back in time because of the simplicity of the Friedmann Eq., since we only need to follow the evolution of a local ‘fair sample’ spherical volume.
Summary: Thermal History